*Slope Fields* **and L'Hopitals Rule** **Name** ______________________________
**AP Calculus AB**

❶ Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

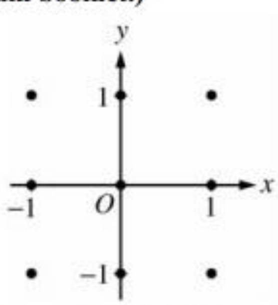
(a) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of $f$ have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left( \frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

❷ Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**




(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of $c$.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

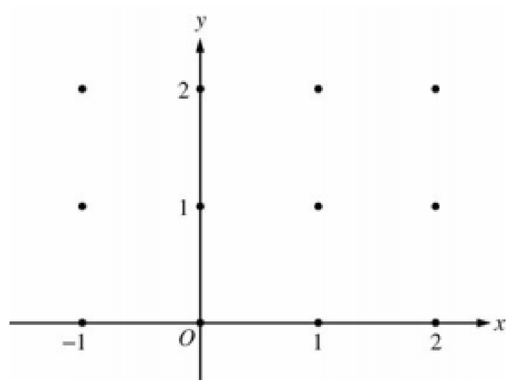
❸ Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of $x$.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

❹ Consider the differential equation $\frac{dy}{dx} = \frac{-xy^2}{2}$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
**(Note: Use the axes provided in the test booklet.)**




(b) Write an equation for the line tangent to the graph of $f$ at $x = -1$.

(c) Find the solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

Use L'Hopitals Rule to evaluate each limit.

1. $\lim_{x \to 0} \frac{\sin 2x}{x}$
2. $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$
3. $\lim_{x \to \infty} \left( xe^{1/x} - x \right)$
4. $\lim_{x \to 0} \frac{e^{5x} - 1}{x}$
5. $\lim_{x \to \infty} \frac{11x + e^{-x}}{7x}$
6. $\lim_{x \to \infty} x^{1/x}$
7. $\lim_{x \to \infty} \frac{5x^2 - 3x - 14}{x^2 + x - 6}$
8. $\lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x$

❶

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$

2 : $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$

$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$

Thus, the graph of $f$ has a relative maximum at the point $(-2, 8)$.

2 : conclusion with justificati

(c) $\lim_{x\to -1}(g(x) - 2) = 0$ and $\lim_{x\to -1} 3(x+1)^2 = 0$

Using L'Hospital's Rule,

$\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right)$

$\lim_{x\to -1} g'(x) = 0$ and $\lim_{x\to -1} 6(x+1) = 0$

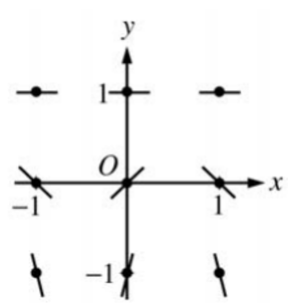
Using L'Hospital's Rule,

$\lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$

3 : $\begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

❷

(a)




2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

1 : $c = 1$

(c) $\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$

$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$

$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$

$1 = \frac{1}{\pi}\sin(\pi) + C = C$

$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$

$\frac{\pi}{1-y} = \sin(\pi x) + \pi$

$y = 1 - \frac{\pi}{\sin(\pi x) + \pi}$ for $-\infty < x < \infty$

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

❸

(b) Since $f$ is differentiable at $x = 1$, $f$ is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1}(x^3 - 1)$ and we may apply L'Hospital's Rule.

$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$$

2 : { 1 : use of L'Hospital's Rule; 1 : answer }

(c) $\frac{dy}{dx} = 1 - y$

$$\int \frac{1}{1 - y}\, dy = \int 1\, dx$$

$$-\ln|1 - y| = x + C$$

$$-\ln 1 = 1 + C \Rightarrow C = -1$$

$$\ln|1 - y| = 1 - x$$

$$|1 - y| = e^{1-x}$$

$$f(x) = 1 - e^{1-x}$$

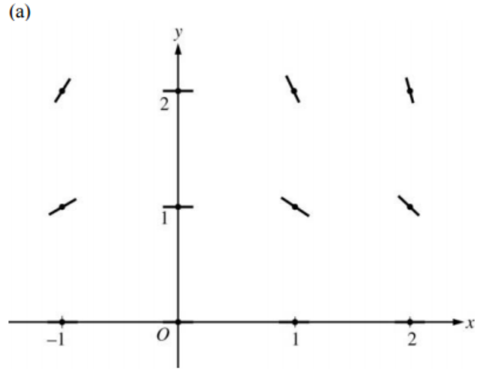
5 : { 1 : separation of variables; 1 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for $y$ }

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

❹

(a)




2 : { 1 : zero slopes; 1 : nonzero slopes }

(b) Slope $= \frac{-(-1)4}{2} = 2$

$$y - 2 = 2(x + 1)$$

1 : equation

(c) $\frac{1}{y^2} dy = -\frac{x}{2} dx$

$$-\frac{1}{y} = -\frac{x^2}{4} + C$$

$$-\frac{1}{2} = -\frac{1}{4} + C;\ C = -\frac{1}{4}$$

$$y = \frac{1}{\frac{x^2}{4} + \frac{1}{4}} = \frac{4}{x^2 + 1}$$

6 : { 1 : separates variables; 2 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for $y$ }

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

Answers

1. 2
2. 6
3. 1
4. 5
5. 11/7
6. 1
7. 5
8. $e^2$